The Maximum Area of a Cyclic (n+2)-gon Theorem

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Abstract

This paper offers the proof of the theorem about the maximum area of a cyclic polygon with one fixed side and n+1 other sides with variable lengths.

Key words: Cyclic Polygons, Maximum Area, Weierstrass Theorem

Introduction

The formulation and proof of this theorem has originated from the solution of Problem #25 in the AMC 12A 2011 test [1].

Theorem

Suppose that a cyclic polygon has the following vertices: A, P_1, P_2,…, P_n, B, where A and B are two points on its circumcircle, and P_1,…, P_n are $n \geq 1$ points on the minor arc AB < π of the same circle. Then the polygon $AP_1,…,P_nB$ in which points P_1,…, P_n divide the arc AB into (n+1) equal arcs occupies the largest area over all such polygons.

The solution described in [1] correctly states and proves for n = 2 that replacing point P_1 on the arc AP_2 by the point $P_1^{'}$ in the middle between points A and P_2 increases the area of polygon AP_1P_2B (if point P_1 was not in the middle between points A and P_2) and, analogously, replacing point P_2 on the arc P_1B by the point $P_2^{'}$ in the middle between points P_1 and B increases the area of polygon AP_1P_2B. These facts, even though correct, do not provide the rigorous proof of the fact that the area of the polygon AP_1P_2B whose vertices P_1 and P_2 divide arc AB into **three** equal arcs is larger than the area of any other polygon AP_1P_2B whose vertices P_1 and P_2 are on the same arc AB.

Proof

We can assume without loss of generality that the circumcircle of a cyclic polygons is a unit circle. Indeed, if radius **r** of the circumcircle is any positive number, then the areas of all its inscribed polygons are proportional to the areas of similar polygons inscribed in a unit circle with the ratio $\mathbf{r^2}$.

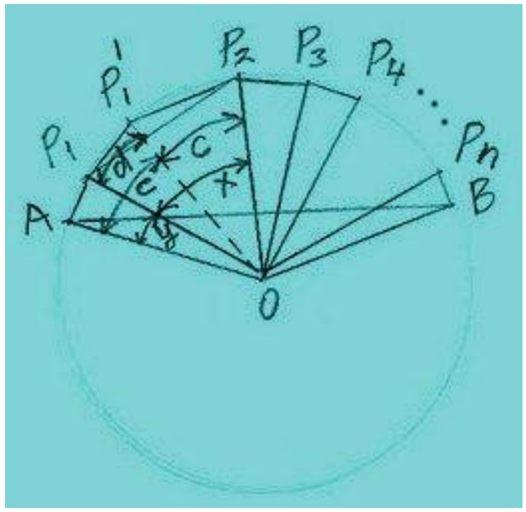

Diagram 1

Let O be the center of the circumcircle of a cyclic polygon $\mathbf{AP_1P_2,\ldots,P_nB}$ and let **S** be the sum of the areas of triangles AOP_1, P_1OP_2, P_2OP_3,…, P_nOB, and $\boldsymbol{S}'$ be the area of the fixed triangle AOB. Then, the area of polygon $\mathbf{AP_1P_2,\ldots,P_nB}$ = $\mathbf{S}$ - $\mathbf{S}'$. Therefore, instead of comparing the areas of any two polygons $\mathbf{AP_1P_2,\ldots,P_nB}$ we can compare the areas of their corresponding polygons $\mathbf{OAP_1P_2,\ldots,P_nB}$ with the same result.

Let any three contiguous vertices (such as **A,** $\mathbf{P_1}$, $\mathbf{P}_2$) of the polygon $\mathbf{AP_1P_2,\ldots,P_nB}$ be the ends of two adjacent arcs measured **x** and **y,** where x > y.

Compare the areas of two polygons: $\mathbf{OAP_1P_2,\ldots,P_nB}$ and the modified polygon in which point $\mathbf{P_1}$ is moved to point $\boldsymbol{P_1'}$ that divides arc $\mathbf{AP}_2$ into two equal arcs with radian measurement c = $\frac{\boldsymbol{\angle AOP_2}}{\mathbf{2}}$.

Denote x = c + d and y = c - d, so that c = $\frac{x+y}{2}$ and d = $\frac{x-y}{2}$ (see diagram 1 above). Notice that $0 < c < \pi$ and $0 < d < \frac{\pi}{2}$ (since $(x - y) < \pi$).

The areas of triangles $\boldsymbol{AOP_1}$ and $\boldsymbol{P_1OP_2}$ are $\frac{1}{2}$*Sin(x) and $\frac{1}{2}$*Sin(y) respectively, since these triangles are isosceles with the side length = 1.

The combined area of these two triangles is:

1/2*(Sin x + Sin y) = 1/2*(Sin(c + d) + sin(c - d)) = 1/2(Sin(c)*Cos(d) + Cos(c)* Sin(d) + Sin(c)*Cos(d) – Cos(c)*Sin(d)) = Sin(c)*Cos(d).

The combined area of the new pair of triangles $\mathbf{A}\boldsymbol{O}\mathbf{P_1'}$ and $\mathbf{P_1'}\boldsymbol{OP_2}$ is:

2* $\frac{1}{2}$*Sin(c) = Sin(c).

Since
$0 < d < \frac{\pi}{2}$; $0 < c < \pi$

it follows that
0 < Cos(d) < 1 and 0 < Sin(c) < 1

and
Sin(c)*Cos(d) < Sin c.

We have proved that the total area of the pair of triangles $\mathbf{AO}\boldsymbol{P}_{\mathbf{1}}'$ and $\boldsymbol{P}_{\mathbf{1}}'\mathbf{O}\boldsymbol{P}_{\mathbf{2}}$ is larger than the total area of the pair of triangles $\mathbf{AO}\boldsymbol{P}_{\mathbf{1}}$ and $\boldsymbol{P}_{\mathbf{1}}\mathbf{OP}_{\mathbf{2}}$.

This also proves that function sin(x) is **strict midpoint concave** in interval [0, π]:

$\frac{\mathbf{sin}(\mathbf{x}) + \mathbf{sin}(\mathbf{y})}{\mathbf{2}} \leq \mathbf{sin}(\frac{x+y}{\mathbf{2}}) = \mathbf{sin}(\mathrm{c})$ with equality achieved only when $\boldsymbol{x} = \boldsymbol{y}$ **.**

Note. It can also be proved geometrically: of the two triangles with the same base $\mathbf{AP_2}$ and altitudes dropped from vertices $\mathbf{P_1}$ and $\mathbf{P_1'}$ respectively, the triangle $\mathbf{AP_1'P_2}$ has larger area than triangle $\mathbf{AP_1P_2}$ since $\mathbf{P_1'}$ is the midpoint of the arc $\mathbf{AP_2}$.

Thus, the only set of points $\mathbf{P_1, P_2,\ldots, P_n}$ for which it is impossible to increase the total area of polygon $\mathbf{OAP_1P_2,\ldots,P_nB}$ by equalizing a pair of adjacent arcs is the one that divides the arc AB into (n+1) equal arcs. We will prove that its area $\frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{n}+\mathbf{1})\boldsymbol{Sin}(\frac{\angle \boldsymbol{AB}}{\boldsymbol{n}+\mathbf{1}})$ is the largest possible area among all polygons $\mathbf{OAP_1P_2,\ldots,P_nB}$**.**

In more general form: the m-dimensional point $(\frac{A}{m}, \frac{A}{m},\ldots, \frac{A}{m})$ yields the maximum for function $\mathbf{F}(X_1, \ldots, \mathbf{Xm}) = \mathbf{Sin}(X_1) + \ldots + \mathbf{Sin}(X_m)$ where **m** ≥ 2; 0 <= Xi <= A;

∀ i: 1 <= i <= m; and $X_1 + X_2 + \ldots + \mathbf{Xm} = \mathbf{A}$**;** A < π.

The domain of function F is the locus of points that belong to one face of an m-dimensional pyramid whose other faces are formed by the axes of coordinates.

Diagram 2 below shows 2-D and 3-D examples of this domain. 2-D domain is the red segment between and including two points: (0, A) and (A, 0).

3-D domain is the triangular face between and including the red boundaries that connect and include points (A, 0, 0), (0, A, 0), and (0, 0, A).

In 2-D case, the equality **x + y = A** holds for all the points and only for the points that belong to the red segment, and, in 3-D case, the equality **x + y + z = A** holds for all the points and only for the points that belong to the triangular face with red boundaries. These coordinates are the lengths of the altitudes of the partial triangles (pyramids), whose total area (volume) is equal to the area (volume) of the entire triangle (pyramid). By analogy, the "slanted" face of an m-dimensional pyramid that includes all its edges and vertices is the locus of points that satisfy the equality $X_1 + X_2 + \dots + \mathbf{Xm} = \mathbf{A}$. It is a closed and bounded region in the m-dimensional space of real numbers.

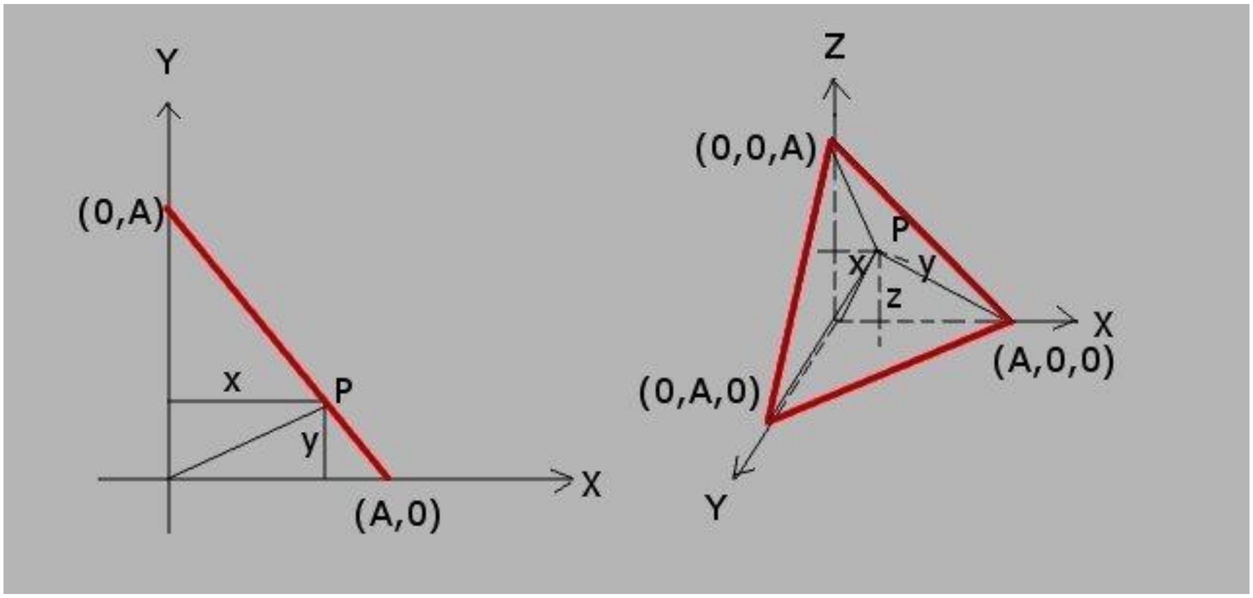

Diagram 2

Since $\mathbf{F}(X_1,\dots, \mathbf{Xm})$ is a continuous real-valued function of **m** real-valued variables in the closed and bounded domain, then, based on Weierstrass theorem, it must have at least one **m**-dimensional point of maximum in its domain.

We have proved earlier that any **m**-dimensional point in the domain of function F that is different from point $(\frac{A}{m}, \frac{A}{m},\dots, \frac{A}{m})$ is NOT the maximum, since it includes at least one pair of adjacent arcs with measurements **x** and **y** ($\mathbf{x} \neq \mathbf{y}$) whose total arc can be divided in two equal arcs with measurements $\frac{x+y}{2}$ that will produce a different **m**-dimensional point in the domain of function F whose value of function F is larger than its value in the previous **m**-dimensional point. Therefore, the only possibility left is that $(\frac{A}{m}, \frac{A}{m},\dots, \frac{A}{m})$ is the single point of maximum of function F in its domain.

Note. Any degenerated cases in which triangles $P_i\mathbf{O}P_{i+1}$ have angle equal to 0 are treated in the same way as normal cases. For example,

if y = 0 and $0 < x < \pi$, then c = $\frac{x}{2}$ and d = $\frac{x}{2}$

and the inequalities

$0 < d < \frac{\pi}{2}$; $0 < c < \pi$
$0 < \text{Cos } d < 1$
$\text{Sin } c * \text{Cos } d < \text{Sin } c$

are still true.

References

[1] Solutions Pamphlet, American Mathematics Competitions, 67th Annual AMC 12 A, February 8, 2011, MAA, Mathematical Association of America.